GENERALIZED FINITE INTEGRAL TRANSFORM

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A generalized finite integral transform combining the Fourier and Hankel transforms is introduced. This transform, together with a Laplace transformation with respect to time, makes possible the simultaneous solution of the problems for a plate, a cylinder, and a sphere.

Finite integral transforms are convenient for solving problems of theoretical physics, when the value of the investigated quantity at the initial instant is given in the form of a function of the coordinates.

The method has been developed in [1–11] and elsewhere. Integral transforms were used in [12] to solve problems of nonsteady heat and mass transfer.

This article introduces a generalized transform that combines the Fourier and Hankel transforms. For this purpose, we employ the functions [13]

$$\Phi_\Gamma(x) = 1 - \frac{x^2}{2(\Gamma+1)} + \frac{x^4}{2.4(\Gamma+1)(\Gamma+3)} - \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!!\,(\Gamma+2m-1)!!}, \quad (1)$$

$$V_\Gamma(x) = \frac{x}{\Gamma+1} - \frac{x^3}{2(\Gamma+1)(\Gamma+3)} + \frac{x^5}{2.4(\Gamma+1)(\Gamma+3)(\Gamma+5)} - \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m)!!\,(\Gamma+2m+1)!!}, \quad (2)$$

which can be expressed in terms of the hypergeometric function

$$\Phi_\Gamma(x) = F\left[\frac{\Gamma+1}{2};\ -\left(\frac{x}{2}\right)^2\right];\ V_\Gamma(x) = \frac{x}{\Gamma+1} F\left[\frac{\Gamma+3}{2};\ -\left(\frac{x}{2}\right)^2\right]. \quad (3)$$

The functions $\Phi_\Gamma(x)$ and $V_\Gamma(x)$ have the great advantage that they make it possible to solve the problems for a plate ($\Gamma = 0$), a cylinder ($\Gamma = 1$), and a sphere ($\Gamma = 2$) simultaneously. For $\Gamma = 0, 1$, and 2, we obtain the usual series defining trigonometric and Bessel functions:

$$\Phi_0(x) = \cos x, \quad \Phi_1(x) = I_0(x), \quad \Phi_2(x) = \frac{\sin x}{x};$$

$$V_0(x) = \sin x, \quad V_1(x) = I_1(x),$$

$$V_2(x) = \frac{\sin x - x\cos x}{x^2}. \quad (4)$$

It is easy to show that, if $\mu_1, \mu_2, \ldots, \mu_n$ are positive roots (numbered in increasing order) of one of the equations

$$\Phi_\Gamma(\mu) = 0, \quad (5)$$

$$V_\Gamma(\mu) = 0, \quad (6)$$

$$\frac{\Phi_\Gamma(\mu)}{V_\Gamma(\mu)} = \frac{\mu}{\mathrm{Bi}}, \quad (7)$$

the functions $\Phi_\Gamma(\mu_1 x), \Phi_\Gamma(\mu_2 x), \ldots, \Phi_\Gamma(\mu_n x)$ form on the interval [0, 1] an orthogonal system with weight x^Γ. Then, for any function $f(x)$ that satisfies the Dirchlet conditions on the interval [0, 1], it is possible to construct a series

$$f(x) = \sum_{n=1}^{\infty} \frac{2\Phi_\Gamma(\mu_n x)}{\Phi_\Gamma^2(\mu_n) + V_\Gamma^2(\mu_n) + \frac{1-\Gamma}{\mu_n}\Phi_\Gamma(\mu_n)V_\Gamma(\mu_n)} \times \int_0^1 x^\Gamma \Phi_\Gamma(\mu_n x) f(x)\,dx. \quad (8)$$

The generalized integral transform is defined as

$$\{f(\mu)\}_\Gamma = \int_0^1 x^\Gamma \Phi_\Gamma(\mu x) f(x)\,dx. \quad (9)$$

Here, μ is a root of one of the three equations (5)–(7). For $\Gamma = 0$ and 1, Eq. (9) gives the Fourier [1] and Hankel [2] transforms as special cases.

Comparing Eq. (8) with definition (9), we see that the integral is exactly $\{f(\mu)\}_\Gamma$. Hence, it follows that the inversion formula for transform (9) has the form

$$f(x) = \sum_{n=1}^{\infty} \frac{2\Phi_\Gamma(\mu_n x)}{\Phi_\Gamma^2(\mu_n) + V_\Gamma^2(\mu_n) + \frac{1-\Gamma}{\mu_n}\Phi_\Gamma(\mu_n)V_\Gamma(\mu_n)} \times \{f(\mu_n)\}_\Gamma. \quad (10)$$

We now apply transform (9) to problems of heat conduction. The temperature field of one-dimensional bodies is described by the equation

$$\frac{\partial\theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2\theta(\xi, \mathrm{Fo})}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial\theta(\xi, \mathrm{Fo})}{\partial\xi} + A\theta(\xi, \mathrm{Fo}) + \mathrm{Po}(\xi, \mathrm{Fo}). \quad (11)$$

The initial temperature is assumed to be a given function of the dimensionless coordinate

$$\theta(\xi, 0) = f(\xi). \quad (12)$$

Moreover, for a plate, by virtue of symmetry,

$$\frac{\partial \theta(0,\ \mathrm{Fo})}{\partial \xi} = 0. \quad (13)$$

Assuming that the operator of transform (9) is commutative with the differentiation operator $\partial/\partial\,\mathrm{Fo}$, after multiplying all the terms of Eq. (11) by $\xi^{\Gamma}\Phi_{\Gamma}(\mu\xi)$ and integrating with respect to ξ from 0 to 1, we obtain

$$\frac{\partial\{\theta(\mu,\ \mathrm{Fo})\}_{\Gamma}}{\partial\,\mathrm{Fo}} = \Phi_{\Gamma}(\mu)\ \frac{\partial\theta(1,\ \mathrm{Fo})}{\partial\xi} + \mu V_{\Gamma}(\mu)\,\theta(1,\ \mathrm{Fo}) - \mu^2\{\theta(\mu,\ \mathrm{Fo})\}_{\Gamma} + A\{\theta(\mu,\ \mathrm{Fo})\}_{\Gamma} + \{\mathrm{Po}(\mu,\ \mathrm{Fo})\}_{\Gamma}. \quad (14)$$

Applying a Laplace transformation to Eq. (14), we obtain

$$\{\bar\theta(\mu,\ s)\}_{\Gamma} = \frac{1}{s+\mu^2-A}\times \left[\int_0^1 \xi^{\Gamma}\Phi_{\Gamma}(\mu\xi)\,f(\xi)\,d\xi + \Phi_{\Gamma}(\mu)\,\frac{\partial\bar\theta(1,\ s)}{\partial\xi} + \mu V_{\Gamma}(\mu)\,\bar\theta(1,\ s) + \{\overline{\mathrm{Po}}(\mu,\ s)\}_{\Gamma}\right]. \quad (15)$$

Going over to the inverse transform with respect to the parameter s, we find

$$\{\theta(\mu,\ \mathrm{Fo})\}_{\Gamma} = \exp[(A-\mu^2)\,\mathrm{Fo}]\times \left\{\int_0^1 \xi^{\Gamma}\Phi_{\Gamma}(\mu\xi)\,f(\xi)\,d\xi + \int_0^{\mathrm{Fo}}\exp[(\mu^2-A)\,\mathrm{Fo}^*]\times \left[\Phi_{\Gamma}(\mu)\,\frac{\partial\theta(1,\ \mathrm{Fo}^*)}{\partial\xi} + \mu V_{\Gamma}(\mu)\,\theta(1,\ \mathrm{Fo}^*) + \{\mathrm{Po}(\mu,\ \mathrm{Fo}^*)\}_{\Gamma}\right]d\,\mathrm{Fo}^*\right\}. \quad (16)$$

Substituting (16) into the inversion formula (10), we obtain

$$\theta(\xi,\ \mathrm{Fo}) = \sum_{n=1}^{\infty}\frac{2\Phi_{\Gamma}(\mu_n\xi)}{\Phi_{\Gamma}^2(\mu_n)+V_{\Gamma}^2(\mu_n)+\frac{1-\Gamma}{\mu_n}\Phi_{\Gamma}(\mu_n)\,V_{\Gamma}(\mu_n)}\times \exp[(A-\mu_n^2)\,\mathrm{Fo}]\left\{\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,f(\xi)\,d\xi + \int_0^{\mathrm{Fo}}\exp[(\mu_n^2-A)\,\mathrm{Fo}^*]\times \left[\Phi_{\Gamma}(\mu_n)\,\frac{\partial\theta(1,\ \mathrm{Fo}^*)}{\partial\xi} + \mu_n V_{\Gamma}(\mu_n)\,\theta(1,\ \mathrm{Fo}^*) + \int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,\mathrm{Po}(\xi,\ \mathrm{Fo}^*)\,d\xi\right]d\,\mathrm{Fo}^*\right\}. \quad (17)$$

In boundary conditions of the first kind, the surface temperature of the body is given as a function of time:

$$\theta(1,\ \mathrm{Fo}) = \varphi(\mathrm{Fo}). \quad (18)$$

In this case, we must assume that μ_n are roots of Eq. (5). Substituting (5) and (18) into (17), we find

$$\theta(\xi,\ \mathrm{Fo}) = \sum_{n=1}^{\infty}\frac{2\Phi_{\Gamma}(\mu_n\xi)}{V_{\Gamma}^2(\mu_n)}\exp[(A-\mu_n^2)\,\mathrm{Fo}]\times \left\{\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,f(\xi)\,d\xi + \mu_n V_{\Gamma}(\mu_n)\times \int_0^{\mathrm{Fo}}\varphi(\mathrm{Fo}^*)\exp[(\mu_n^2-A)\,\mathrm{Fo}^*]\,d\,\mathrm{Fo}^* + \int_0^{\mathrm{Fo}}\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,\mathrm{Po}(\xi,\ \mathrm{Fo}^*)\times \exp[(\mu_n^2-A)\,\mathrm{Fo}^*]\,d\xi\,d\,\mathrm{Fo}^*\right\}. \quad (19)$$

In boundary conditions of the second kind, the heat flux at the surface is given as a function of time:

$$\frac{\partial\theta(1,\ \mathrm{Fo})}{\partial\xi} = \mathrm{Ki}(\mathrm{Fo}). \quad (20)$$

In this case, it is necessary to assume that μ_n are roots of Eq. (6). Substituting (6) and (20) into (17), and keeping in mind that $\mu = 0$ is also a root of Eq. (6), we find

$$\theta(\xi,\ \mathrm{Fo}) = (\Gamma+1)\exp(AF)\times \left\{\int_0^1\xi^{\Gamma}f(\xi)\,d\xi + \int_0^{\mathrm{Fo}}\mathrm{Ki}(\mathrm{Fo}^*)\exp(-A\,\mathrm{Fo}^*)\,d\,\mathrm{Fo}^* + \int_0^{\mathrm{Fo}}\int_0^1\xi^{\Gamma}\mathrm{Po}(\xi,\ \mathrm{Fo}^*)\exp(-A\,\mathrm{Fo}^*)\,d\,\mathrm{Fo}^*\right\} + \sum_{n=1}^{\infty}\frac{2\Phi_{\Gamma}(\mu_n\xi)}{\Phi_{\Gamma}^2(\mu_n)}\exp[(A-\mu_n^2)\,\mathrm{Fo}]\times \left\{\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,f(\xi)\,d\xi + \Phi_{\Gamma}(\mu_n)\times \int_0^{\mathrm{Fo}}\mathrm{Ki}(\mathrm{Fo}^*)\exp[(\mu_n^2-A)\,\mathrm{Fo}^*]\,d\,\mathrm{Fo}^* + \int_0^{\mathrm{Fo}}\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,\mathrm{Po}(\xi,\ \mathrm{Fo}^*)\times \exp[(\mu_n^2-A)\,\mathrm{Fo}^*]\,d\xi\,d\,\mathrm{Fo}^*\right\}. \quad (21)$$

In boundary conditions of the third kind, heat exchange with the surrounding medium proceeds according to Newton's law

$$\frac{\partial\theta(1,\ \mathrm{Fo})}{\partial\xi} + \mathrm{Bi}[\theta(1,\ \mathrm{Fo}) - \theta_f(\mathrm{Fo})] = 0, \quad (22)$$

where the temperature of the surrounding medium, $\theta_f(\mathrm{Fo})$, is a given function of time.

In this case, we must assume that μ_n are roots of Eq. (7). Using (7) and (22), from (17) we find

$$\theta(\xi,\ \mathrm{Fo}) = \sum_{n=1}^{\infty}\frac{2\Phi_{\Gamma}(\mu_n\xi)}{V_{\Gamma}^2(\mu_n)}\ \frac{\mathrm{Bi}^2}{\mathrm{Bi}^2+\mu_n^2+(1-\Gamma)\,\mathrm{Bi}}\times \exp[(A-\mu_n^2)\,\mathrm{Fo}]\left\{\int_0^1\xi^{\Gamma}\Phi_{\Gamma}(\mu_n\xi)\,f(\xi)\,d\xi + \mu_n V_{\Gamma}(\mu_n)\times\right.$$

$$\times \int_0^{Fo} \theta_f (Fo^*) \exp [(\mu_n^2 - A) Fo^*] \, d Fo^* +$$

$$+ \int_0^{Fo} \int_0^1 \xi^\Gamma \Phi_\Gamma (\mu_n \xi) \, Po (\xi, Fo^*) \times$$

$$\times \exp [(\mu_n^2 - A) Fo^*] \, d\xi \, d Fo^* \Big\} . \quad (23)$$

If, into the solutions obtained, we substitute the values of the functions $\Phi_\Gamma(x)$ and $V_\Gamma(x)$ in accordance with (4), we obtain the solutions for a plate, a cylinder, and a sphere. From (19), (21), and (23), there follows the series of particular solutions given in [12, 14, 15] and elsewhere.

The uniformity of the equations obtained facilitates programming and computer calculations. For this purpose, it is desirable to compile standard routines for computing $\Phi_\Gamma(x)$ and $V_\Gamma(x)$.

The proposed integral transform can easily be used to solve a number of problems of thereotical physics, and also the system of equations of heat and mass transfer given in [12].

NOTATION

x is the independent variable; Γ is a constant equal to 0, 1, and 2, respectively, for a plate, a cylinder, and a sphere; $\Phi_\Gamma(x)$ is a function defined by Eq. (1); $V_\Gamma(x)$ is a function defined by Eq. (2); ξ is a dimensionless coordinate; Fo is the Fourier number; $\theta(\xi, Fo)$ is the dimensionless temperature; $Po(\xi, Fo)$ is the Pomerantsev number; A is a dimensionless parameter; $\varphi(Fo)$ is the dimensionless surface temperature; Ki(Fo) is the Kirpichev number; Bi is the Biot number; $\theta_f(Fo)$ is the dimensionless temperature of the surrounding medium; μ_n are the roots of one of the three equations (5)–(7); s is the Laplace transform parameter.

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